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周期边界条件下 Sturm-Liouville 方程组特征值问题的迹公式

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摘要：讨论了周期边界条件下 Sturm-Liouville 方程组特征值问题.首先将特征值的存在性问题化为一个整函数零点的存在性问题，然后借助于一个积分恒等式，采用留数方法，得到了周期边界条件下 Sturm-Liouville 方程组特征值问题的迹公式.

关键词：周期边界条件；特征值；渐近估计；迹公式

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0 引言

微分算子特征值的迹公式深刻揭示了微分算子的谱结构，在特征值的计算、反问题以及孤子理论和可积系统理论中都有重要作用.众所周知，矩阵的迹是指所有特征值的和，它等于对角线上各元素之和，然而矩阵的单个特征值是比较难求的.对于微分算子的迹，是否也能用算子量直接表示呢？Gelfand 和 Levitan 在1953年获得如下的 Sturm-Liouville 问题的迹公式：

$$-y''+q(x)y=\lambda y,\ y'(0)=y'(\pi)=0,$$

特征值的迹公式为

$$\sum_{n=0}^{\infty}\left(\lambda_n-n^2-\frac{1}{\pi}\int_0^{\pi}q(x)\mathrm{d}x\right)=\frac{q(0)+q(\pi)}{4}-\frac{1}{2\pi}\int_0^{\pi}q(x)\mathrm{d}x.$$

随后，关于迹公式的研究出现了一系列的报道[1-4].

本文采用留数方法获得了如下问题的迹公式：

$$(\mathrm{E})\quad\begin{cases}Ly=\lambda y,\\ y(0)=y(\pi),\\ y'(0)=y'(\pi).\end{cases}$$

其中，

$$L=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\begin{pmatrix}u(x)&w(x)\\w(x)&v(x)\end{pmatrix},\ y=\begin{pmatrix}y_1\\y_2\end{pmatrix},\ x\in[0,\pi],\ u(x),w(x),v(x)\in\mathbf{C}^2[0,\pi].$$

1 Sturm-Liouville 方程组 Cauchy 问题解的渐近估计

考虑如下的 Cauchy 问题：

(c_1) $L\phi=\lambda\phi,\ \phi_1(0,\lambda)=1,\ \phi_1'(0,\lambda)=0,\ \phi_2(0,\lambda)=0,\ \phi_2'(0,\lambda)=0;$

(c_2) $L\varphi=\lambda\varphi,\ \varphi_1(0,\lambda)=0,\ \varphi_1'(0,\lambda)=-1,\ \varphi_2(0,\lambda)=0,\ \varphi_2'(0,\lambda)=0;$

(c_3) $L\chi=\lambda\chi,\ \chi_1(0,\lambda)=0,\ \chi_1'(0,\lambda)=0,\ \chi_2(0,\lambda)=1,\ \chi_2'(0,\lambda)=0;$

(c_4) $L\Psi=\lambda\Psi,\ \Psi_1(0,\lambda)=0,\ \Psi_1'(0,\lambda)=0,\ \Psi_2(0,\lambda)=0,\ \Psi_2'(0,\lambda)=-1;$

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显然Wronski行列式$W[\phi,\varphi,\chi,\Psi]=1$，4个解线性无关，可以构成一个基础解系．

由常数变异法可得引理1．

引理1 初值问题$\{y''+s^2y=f(x),y(a)=A,y'(a)=B\}$的解为

$$y(x)=A\cos s(x-a)+\frac{1}{2}B\sin s(x-a)+\int_a^x\frac{\sin s(x-\xi)}{s}f(\xi)\mathrm{d}\xi.$$

命题1 记$\lambda=s^2$，$s=\sigma+\mathrm{i}y\tau$，则当$\lambda\to\infty$，下面的渐近式在$0\leqslant x\leqslant\pi$上一致成立：

$$\begin{cases}\phi_1(x,\lambda)=\cos sx+\dfrac{1}{s}k_1(x)\sin sx+\dfrac{1}{s^2}k_4(x)\cos sx+O\left(\dfrac{1}{s^3}\mathrm{e}^{|\tau|x}\right),\\ \phi_2(x,\lambda)=\dfrac{1}{s}k_2(x)\sin sx+\dfrac{1}{s^2}k_{10}(x)\cos sx+\dfrac{1}{s^3}k_{12}(x)\sin sx+O\left(\dfrac{1}{s^4}\mathrm{e}^{|\tau|x}\right),\\ \phi_1'(x,\lambda)=-s\sin sx+k_1(x)\cos sx+\dfrac{1}{s}[\mathrm{k}_1'(x)-k_4(x)]\sin sx+O\left(\dfrac{1}{s^2}\mathrm{e}^{|\tau|x}\right),\\ \phi_2'(x,\lambda)=k_2(x)\cos sx+\dfrac{1}{s}[\mathrm{k}_2'(x)-k_{10}(x)]\sin sx+\dfrac{1}{s^2}[\mathrm{k}_{10}'(x)+k_{12}(x)]\cos sx+O\left(\dfrac{1}{s^3}\mathrm{e}^{|\tau|x}\right);\end{cases}$$

$$\begin{cases}\varphi_1(x,\lambda)=-\dfrac{1}{s}\sin sx+\dfrac{1}{s^2}k_1(x)\cos sx+\dfrac{1}{s^3}k_8(x)\sin sx+O\left(\dfrac{1}{s^4}\mathrm{e}^{|\tau|x}\right),\\ \varphi_2(x,\lambda)=\dfrac{1}{s^2}k_2(x)\cos sx+\dfrac{1}{s^3}k_9(x)\sin sx+O\left(\dfrac{1}{s^4}\mathrm{e}^{|\tau|x}\right),\\ \varphi_1'(x,\lambda)=-\cos sx-\dfrac{1}{s}k_1(x)\sin sx+\dfrac{1}{s}[\mathrm{k}_1'(x)+k_8(x)]\cos sx+O\left(\dfrac{1}{s^3}\mathrm{e}^{|\tau|x}\right),\\ \varphi_2'(x,\lambda)=-\dfrac{1}{s}k_2(x)\sin sx+\dfrac{1}{s^2}[\mathrm{k}_2'(x)+k_9(x)]\cos sx+O\left(\dfrac{1}{s^3}\mathrm{e}^{|\tau|x}\right);\end{cases}$$

$$\begin{cases}\chi_1(x,\lambda)=\dfrac{1}{s}k_2(x)\sin sx+\dfrac{1}{s^2}k_6(x)\cos sx+\dfrac{1}{s^3}k_{13}(x)\sin sx+O\left(\dfrac{1}{s^4}\mathrm{e}^{|\tau|x}\right),\\ \chi_2(x,\lambda)=\cos sx+\dfrac{1}{s}k_3(x)\sin sx+\dfrac{1}{s^2}k_7(x)\cos sx+O\left(\dfrac{1}{s^3}\mathrm{e}^{|\tau|x}\right),\\ \chi_1'(x,\lambda)=k_2(x)\cos sx-\dfrac{1}{s}[\mathrm{k}_2'(x)-k_6(x)]\sin sx+\dfrac{1}{s^2}[\mathrm{k}_6'(x)+k_{13}(x)]\cos sx+O\left(\dfrac{1}{s^3}\mathrm{e}^{|\tau|x}\right),\\ \chi_2'(x,\lambda)=-s\sin sx+k_3(x)\cos sx+\dfrac{1}{s}[\mathrm{k}_3'(x)-k_7(x)]\sin sx+O\left(\dfrac{1}{s^2}\mathrm{e}^{|\tau|x}\right);\end{cases}$$

$$\begin{cases}\Psi_1(x,\lambda)=\dfrac{1}{s^2}k_2(x)\cos sx+\dfrac{1}{s^3}k_{11}(x)\sin sx+O\left(\dfrac{1}{s^4}\mathrm{e}^{|\tau|x}\right),\\ \Psi_2(x,\lambda)=-\dfrac{1}{s}\sin sx+\dfrac{1}{s^2}k_3(x)\cos sx+\dfrac{1}{s^3}k_5(x)\sin sx+O\left(\dfrac{1}{s^4}\mathrm{e}^{|\tau|x}\right),\\ \Psi_1'(x,\lambda)=-\dfrac{1}{s}k_2(x)\sin sx+\dfrac{1}{s^2}[\mathrm{k}_2'(x)+k_{11}(x)]\cos sx+O\left(\dfrac{1}{s^3}\mathrm{e}^{|\tau|x}\right),\\ \Psi_2'(x,\lambda)=-\cos sx-\dfrac{1}{s}k_3(x)\sin sx+\dfrac{1}{s}[\mathrm{k}_3'(x)+k_5(x)]\cos sx+O\left(\dfrac{1}{s^3}\mathrm{e}^{|\tau|x}\right).\end{cases}$$

其中，

$$k_1(x)=\frac{1}{2}\int_0^x u(\xi)\mathrm{d}\xi,\ k_2(x)=\frac{1}{2}\int_0^x w(\xi)\mathrm{d}\xi,\ k_3(x)=\frac{1}{2}\int_0^x v(\xi)\mathrm{d}\xi,$$

$$k_4(x)=\frac{1}{4}[\mathrm{u}(\mathrm{x})-\mathrm{u}(0)]-\frac{1}{2}[\mathrm{k}_1^2(x)+k_2^2(x)],$$

$$k_5(x)=-\frac{1}{4}[\mathrm{v}(\mathrm{x})+\mathrm{v}(0)]+\frac{1}{2}\int_0^x[\omega(\xi)\mathrm{k}_2(\xi)+v(\xi)k_4(\xi)]\mathrm{d}\xi,$$

$$k_6(x)=\frac{1}{4}[\mathrm{w}(\mathrm{x})-\mathrm{w}(0)]-\frac{1}{2}\int_0^x[\mathrm{u}(\xi)\mathrm{k}_2(\xi)+w(\xi)k_3(\xi)]\mathrm{d}\xi,$$

$$k_7(x)=\frac{1}{4}[\mathrm{v}(\mathrm{x})-\mathrm{v}(0)]-\frac{1}{2}\int_0^x[\mathrm{w}(\xi)\mathrm{k}_2(\xi)+v(\xi)k_3(\xi)]\mathrm{d}\xi,$$

$$k_8(x)=-\frac{1}{4}[u(x)+u(0)]+\frac{1}{2}[k_1^2(x)+k_2^2(x)],$$

$$k_9(x)=-\frac{1}{4}[w(x)+w(0)]+\frac{1}{2}\int_0^x[w(\xi)k_1(\xi)+v(\xi)k_2(\xi)]\mathrm{d}\xi,$$

$$k_{10}(x)=\frac{1}{4}[w(x)-w(0)]-\frac{1}{2}\int_0^x[w(\xi)k_1(\xi)+v(\xi)k_2(\xi)]\mathrm{d}\xi,$$

$$k_{11}(x)=-\frac{1}{4}[w(x)+w(0)]+\frac{1}{2}\int_0^x[u(\xi)k_2(\xi)+w(\xi)k_3(\xi)]\mathrm{d}\xi,$$

$$k_{12}(x)=-\frac{1}{8}[w'(x)+w'(0)]+\frac{1}{4}[w(\xi)k_1(\xi)+v(\xi)k_2(\xi)+w(0)k_1(0)+v(0)k_2(0)]+\frac{1}{2}\int_0^x[w(\xi)k_4(\xi)+v(\xi)k_{10}(\xi)]\mathrm{d}\xi,$$

$$k_{13}(x)=-\frac{1}{8}[w'(x)+w'(0)]+\frac{1}{4}[u(\xi)k_2(\xi)+w(\xi)k_3(\xi)+u(0)k_2(0)+w(0)k_3(0)]+\frac{1}{2}\int_0^x[u(\xi)k_6(\xi)+w(\xi)k_7(\xi)]\mathrm{d}\xi.$$

证明 以 Cauchy 问题(c_2)解的渐近式为例进行证明，其余类似可证．(c_2)即

$$\begin{cases}\varphi_1''+s^2\varphi_1=u\varphi_1+w\varphi_2,\ \varphi_1(0)=0,\ \varphi_1'(0)=-1;\\ \varphi_2''+s^2\varphi_2=w\varphi_1+v\varphi_2,\ \varphi_2(0)=0,\ \varphi_2'(0)=0.\end{cases}$$

由引理1得

$$\begin{cases}\varphi_1=-\dfrac{1}{s}\sin sx+\displaystyle\int_0^x\frac{\sin s(x-\xi)}{s}[u(\xi)\varphi_1(\xi)+w(\xi)\varphi_2(\xi)]\mathrm{d}\xi,\\ \varphi_2=\displaystyle\int_0^x\frac{\sin s(x-\xi)}{s}[w(\xi)\varphi_1(\xi)+v(\xi)\varphi_2(\xi)]\mathrm{d}\xi.\end{cases}\tag{1}$$

令 $\varphi_1=f_1\mathrm{e}^{|\tau|x}$，$\varphi_2=f_2\mathrm{e}^{|\tau|x}$，因为 $|\sin sx|\leqslant\mathrm{e}^{|\tau|x}$，$|\sin s(x-\xi)|\leqslant\mathrm{e}^{|\tau|(x-\xi)}$，代入(1)式右端得

$$\begin{cases}|f_1|\leqslant\dfrac{1}{s}+\displaystyle\int_0^x\frac{1}{|s|}[|u(\xi)|\cdot|f_1|+|w(\xi)|\cdot|f_2|]\mathrm{d}\xi,\\ |f_2|\leqslant\displaystyle\int_0^x\frac{1}{|s|}[|w(\xi)|\cdot|f_1|+|v(\xi)|\cdot|f_2|]\mathrm{d}\xi.\end{cases}$$

设 $M=\max\{|u(\xi)|,|w(\xi)|,|v(\xi)|\}$，由 Gronwall 不等式可得

$$|f_1|+|f_2|\leqslant\frac{1}{s}\mathrm{e}^{\frac{2M\pi}{|s|}}.$$

即 $f_1=O\left(\dfrac{1}{|s|}\right)$，$f_2=O\left(\dfrac{1}{|s|}\right)$．所以

$$\begin{cases}\varphi_1=O\left(\dfrac{1}{|s|}\mathrm{e}^{|\tau|x}\right),\\ \varphi_2=O\left(\dfrac{1}{|s|}\mathrm{e}^{|\tau|x}\right).\end{cases}$$

把所得 φ_1，φ_2 代入(1)式，再经反复迭代可得

$$\begin{cases}\varphi_1(x,\lambda)=-\dfrac{1}{s}\sin sx+\dfrac{1}{s^2}k_1(x)\cos sx+\dfrac{1}{s^3}k_8(x)\sin sx+O\left(\dfrac{1}{s^4}\mathrm{e}^{|\tau|x}\right),\\ \varphi_2(x,\lambda)=\dfrac{1}{s^2}k_2(x)\cos sx+\dfrac{1}{s^3}k_9(x)\sin sx+O\left(\dfrac{1}{s^4}\mathrm{e}^{|\tau|x}\right).\end{cases}$$

同理可证 ϕ_1，ϕ_2，χ_1，χ_2，Ψ_1，Ψ_2 的渐近式．

下面再以 $\varphi_1'(x,y)$ 的渐近式为例进行证明，其余类似可证．

由 $\varphi_1=-\dfrac{1}{s}\sin sx+\displaystyle\int_0^x\frac{\sin s(x-\xi)}{s}[u(\xi)\varphi_1(\xi)+w(\xi)\varphi_2(\xi)]\mathrm{d}\xi$，两边对 x 求导得

$$\varphi_1'=-\cos sx+\int_0^x\cos s(x-\xi)[u(\xi)\varphi_1(\xi)+w(\xi)\varphi_2(\xi)]\mathrm{d}\xi.$$

把 φ_1，φ_2 的渐近式代入得

$$\varphi_1'(x,\lambda)=-\cos sx-\frac{1}{s}k_1(x)\sin sx+\frac{1}{s}[\mathrm{k}_1'(x)+k_8(x)]\cos sx+O\left(\frac{1}{s^3}\mathrm{e}^{|\tau|x}\right).$$

同理可证φ_2'，ϕ_1'，ϕ_2'，χ_1'，χ_2'，Ψ_1'，Ψ_2'的渐近式.

2　特征值问题的迹公式

命题2　设

$$\omega(\lambda)=\begin{vmatrix}\phi_1(\pi,\lambda)-1 & \varphi_1(\pi,\lambda) & \chi_1(\pi,\lambda) & \Psi_1(\pi,\lambda)\\ \phi_1'(\pi,\lambda) & \varphi_1'(\pi,\lambda)+1 & \chi_1'(\pi,\lambda) & \Psi_1'(\pi,\lambda)\\ \phi_2(\pi,\lambda) & \varphi_2(\pi,\lambda) & \chi_2(\pi,\lambda)-1 & \Psi_2(\pi,\lambda)\\ \phi_2'(\pi,\lambda) & \varphi_2'(\pi,\lambda) & \chi_2'(\pi,\lambda) & \Psi_2'(\pi,\lambda)+1\end{vmatrix},$$

则问题(E)的特征值集合与$\omega(\lambda)$的零点集合重合.

证明　令$y(x,\lambda)=c_1\phi(x,\lambda)+c_2\varphi(x,\lambda)+c_3\chi(x,\lambda)+c_4\Psi(x,\lambda)$，则$y(x,\lambda)$是问题(E)的通解.设$\lambda$是特征值.由(E)的周期边界条件得

$$\begin{cases}c_1\phi_1(0,\lambda)+c_2\varphi_1(0,\lambda)+c_3\chi_1(0,\lambda)+c_4\Psi_1(0,\lambda)=c_1\phi_1(\pi,\lambda)+c_2\varphi_1(\pi,\lambda)+c_3\chi_1(\pi,\lambda)+c_4\Psi_1(\pi,\lambda),\\ c_1\phi_2(0,\lambda)+c_2\varphi_2(0,\lambda)+c_3\chi_2(0,\lambda)+c_4\Psi_2(0,\lambda)=c_1\phi_2(\pi,\lambda)+c_2\varphi_2(\pi,\lambda)+c_3\chi_2(\pi,\lambda)+c_4\Psi_2(\pi,\lambda),\\ c_1\phi_1'(0,\lambda)+c_2\varphi_1'(0,\lambda)+c_3\chi_1'(0,\lambda)+c_4\Psi_1'(0,\lambda)=c_1\phi_1'(\pi,\lambda)+c_2\varphi_1'(\pi,\lambda)+c_3\chi_1'(\pi,\lambda)+c_4\Psi_1'(\pi,\lambda),\\ c_1\phi_2'(0,\lambda)+c_2\varphi_2'(0,\lambda)+c_3\chi_2'(0,\lambda)+c_4\Psi_2'(0,\lambda)=c_1\phi_2'(\pi,\lambda)+c_2\varphi_2'(\pi,\lambda)+c_3\chi_2'(\pi,\lambda)+c_4\Psi_2'(\pi,\lambda).\end{cases}$$

把Cauchy条件代入上式得

$$\begin{cases}c_1[\phi_1(\pi,\lambda)-1]+c_2\varphi_1(\pi,\lambda)+c_3\chi_1(\pi,\lambda)+c_4\psi_1(\pi,\lambda)=0,\\ c_1\phi_1'(\pi,\lambda)+c_2[\varphi_1'(\pi,\lambda)+1]+c_3\chi_1'(\pi,\lambda)+c_4\psi_1'(\pi,\lambda)=0,\\ c_1\phi_2(\pi,\lambda)+c_2\varphi_2(\pi,\lambda)+c_3[\chi_2(\pi,\lambda)-1]+c_4\psi_2(\pi,\lambda)=0,\\ c_1\phi_2'(\pi,\lambda)+c_2\varphi_2'(\pi,\lambda)+c_3\chi_2'(\pi,\lambda)+c_4[\psi_2'(\pi,\lambda)+1]=0.\end{cases}$$

因为c_1，c_2，c_3，c_4不全为零，故有

$$\begin{vmatrix}\phi_1(\pi,\lambda)-1 & \varphi_1(\pi,\lambda) & \chi_1(\pi,\lambda) & \Psi_1(\pi,\lambda)\\ \phi_1'(\pi,\lambda) & \varphi_1'(\pi,\lambda)+1 & \chi_1'(\pi,\lambda) & \Psi_1'(\pi,\lambda)\\ \phi_2(\pi,\lambda)-1 & \varphi_2(\pi,\lambda) & \chi_2(\pi,\lambda)-1 & \Psi_2(\pi,\lambda)\\ \phi_2'(\pi,\lambda) & \varphi_2'(\pi,\lambda)+1 & \chi_2'(\pi,\lambda) & \Psi_2'(\pi,\lambda)+1\end{vmatrix}=0.$$

即问题(E)的特征值集合与$\omega(\lambda)$的零点集合重合.

引理2　当$\sigma=2N+1$，$|\tau|\geqslant\frac{1}{\pi}$时，$\left|\frac{\mathrm{e}^{|\tau|\pi}}{\sin^4\frac{s\pi}{2}}\right|$有界.

证明　易知$|\sin(x+\mathrm{i}y)|^2=\sin^2x+\mathrm{sh}^2x$.

当$\sigma=2N+1$时，$\left|\sin\frac{s\pi}{2}\right|^2=\left|\sin\frac{(\sigma+\mathrm{i}\tau)\pi}{2}\right|^2=\mathrm{ch}^2\frac{\tau\pi}{2}$，所以$\left|\frac{\mathrm{e}^{|\tau|\pi}}{\sin^4\left(\frac{s\pi}{2}\right)}\right|=\frac{\mathrm{e}^{|\tau|\pi}}{\mathrm{ch}^4\frac{\tau\pi}{2}}\leqslant 8$.

当$|\tau|\geqslant\frac{1}{\pi}$时，$\left|\sin\frac{s\pi}{2}\right|^2=\left|\sin\frac{(\sigma+\mathrm{i}\tau)\pi}{2}\right|^2\geqslant\mathrm{sh}^2\frac{\tau\pi}{2}$，所以$\left|\frac{\mathrm{e}^{|\tau|\pi}}{\sin^4\left(\frac{s\pi}{2}\right)}\right|\leqslant\frac{\mathrm{e}^{|\tau|\pi}}{\mathrm{sh}^4\frac{\tau\pi}{2}}\leqslant 128$.

综上可知：当$\sigma=2N+1$，$|\tau|\geqslant\frac{1}{\pi}$时，$\left|\frac{\mathrm{e}^{|\tau|\pi}}{\sin^4\left(\frac{s\pi}{2}\right)}\right|$有界.

将$\omega(\lambda)$的行列式展开，并把ϕ_1，ϕ_2，φ_1，φ_2，χ_1，χ_2，Ψ_1，Ψ_2的渐近式和ϕ_1'，ϕ_2'，φ_1'，φ_2'，χ_1'，χ_2'，Ψ_1'，Ψ_2'的渐近式代入，经过计算可得

$$\omega(\lambda)=16\sin^4\left(\frac{s\pi}{2}\right)+\frac{16p_1}{s}(\sin s\pi\cos s\pi-\sin s\pi)+\frac{16}{s^2}(p_2+p_3\cos s\pi+p_4\cos^2 s\pi+p_5\sin^2 s\pi)+O\left(\frac{1}{s^3}\mathrm{e}^{|\tau|x}\right).$$

其中，

$$p_1=\frac{1}{4}[k_1(\pi)+k_3(\pi)],$$

$$p_2=\frac{1}{8}[k_1^2(\pi)+k_2^2(\pi)+k_3^2(\pi)+k_4(\pi)+k_7(\pi)-k_5(\pi)-k_8(\pi)-k_1'(\pi)-k_3'(\pi)],$$

$$p_3=\frac{1}{8}[2k_5(\pi)+2k_8(\pi)+2k_1'(\pi)+2k_3'(\pi)-2k_4(\pi)-2k_7(\pi)-k_1^2(\pi)-2k_2^2(\pi)-k_3^2(\pi)],$$

$$p_4=\frac{1}{8}[k_4(\pi)+k_7(\pi)+k_2^2(\pi)-k_5(\pi)-k_8(\pi)-k_1'(\pi)-k_3'(\pi)],$$

$$p_5=\frac{1}{8}[2k_1(\pi)k_3(\pi)-k_2^2(\pi)].$$

记 $\Omega(\lambda)=16\sin^4\left(\frac{s\pi}{2}\right)$，则有

$$\frac{\omega(\lambda)}{\Omega(\lambda)}=1+\frac{1}{s}\cdot\frac{p_1\sin s\pi(\cos s\pi-1)}{\sin^4\left(\frac{s\pi}{2}\right)}+\frac{1}{s^2}\cdot\frac{p_2+p_3\cos s\pi+p_4\cos^2 s\pi+p_5\sin^2 s\pi}{\sin^4\left(\frac{s\pi}{2}\right)}+O\left(\frac{\mathrm{e}^{|\tau|\pi}}{s^3\sin^4\frac{s\pi}{2}}\right).$$

命题3 当 N 充分大时，$\omega(\lambda)$与 $\Omega(\lambda)$在 C_N 内有相同多的零点.

在 s 平面上取折线 L_N，其中，$A=2N+1$，$B=(2N+1)(1+\mathrm{i})$，$C=(2N+1)(-1+\mathrm{i})$，$D=-(2N+1)$. 当 $\sigma=R_N=2N+1$ 时，折线 L_N 即是从 R_N 出发，经$(1+\mathrm{i})R_N$，$(-1+\mathrm{i})R_N$，到达$-R_N$. 变换 $\lambda=s^2$ 把 s 平面上折线 L_N 变成了 λ 平面上的单闭迴路 C_N，它是由 $\lambda=p_1+\mathrm{i}p_2$ 平面上的 2 段抛物线弧组成，抛物线方程分别是$\pm p_1=R_N^2-\left(\frac{p_2}{2R_N}\right)^2$. 当 N 充分大时，在迴路 C_N 上$\left|\frac{\omega(\lambda)}{\Omega(\lambda)}-1\right|<1$，由 Rouche 定理[5]，由于 $\Omega(\lambda)=16\sin^4\left(\frac{s\pi}{2}\right)$ 在 C_N 内有 4 重零点，故 $\omega(\lambda)$在 C_N 内有 4 个零点，分别记为 λ_{1_n}，λ_{2_n}，λ_{3_n}，λ_{4_n}.

定理1 对于问题(E)则有迹公式

$$\sum_{-N}^{N}\left(\lambda_{1_n}+\lambda_{2_n}+\lambda_{3_n}+\lambda_{4_n}-16n^2-\frac{16p_1}{\pi}\right)=-4p_4-\frac{8p_1}{\pi}+O\left(\frac{1}{N}\right).$$

其中，

$$p_1=\frac{1}{4}[k_1(\pi)+k_3(\pi)],$$

$$p_4=\frac{1}{8}[k_4(\pi)+k_7(\pi)+k_2^2(\pi)-k_5(\pi)-k_8(\pi)-k_1'(\pi)-k_3'(\pi)].$$

证明 由命题 3 知，当 N 充分大时，$\omega(\lambda)$与 $\Omega(\lambda)$在 C_N 内有相同多的零点，$\ln\frac{\omega(\lambda)}{\Omega(\lambda)}$在 C_N 上为单值解析函数，对下面的恒等式沿 C_N 作迴路积分，并除以 $2\pi\mathrm{i}$，有

$$\lambda\left[\frac{\omega'(\lambda)}{\omega(\lambda)}-\frac{\Omega'(\lambda)}{\Omega(\lambda)}\right]=-\ln\frac{\omega(\lambda)}{\Omega(\lambda)}+\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\lambda\ln\frac{\omega(\lambda)}{\Omega(\lambda)}\right].$$

由于$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\lambda\ln\frac{\omega(\lambda)}{\Omega(\lambda)}\right]$单值解析，所以在 C_N 上积分为零. 即

$$\sum_{-N}^{N}[\lambda_{1_n}+\lambda_{2_n}+\lambda_{3_n}+\lambda_{4_n}-4(2n)^2]=-\frac{1}{2\pi\mathrm{i}}\oint_{C_N}\ln\frac{\omega(\lambda)}{\Omega(\lambda)}\mathrm{d}\lambda,$$

通过留数计算得

$$\sum_{-N}^{N}\left[\lambda_{1_n}+\lambda_{2_n}+\lambda_{3_n}+\lambda_{4_n}-16n^2-\frac{16p_1}{\pi}\right]=-4p_4-\frac{8p_1}{\pi}+O\left(\frac{1}{N}\right).$$

参考文献：

[1] 李梦如．带有非定局边界条件的 Sturm-Liouville 算子的迹[J]．郑州大学学报：自然科学版，1981，13(2)：9-15．

[2] 李梦如．离散 Sturm-Liouville 问题特征值的迹公式[J]．数学物理学报，1992，12(3)：303-307．

[3] Wu Yongtang，Li Mengru．Some trace formulas for the eigenvalue problem of Schrödinger system[J]．Phys Soc Jpn，1999，68：1107-1114．

[4] 曹策问．微分算子的迹[J]．数学进展，1989，18(2)：170-178．

[5] 钟玉泉．复变函数论[M]．北京：高等教育出版社，2004．

Trace Formula for the Eigenvalue Problem of Sturm-Liouville Equation System with Periodic Boundary Condition

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Abstract：The eigenvalue problem of Sturm-Liouville equation system with periodic boundary condition is discussed．The eigenvalue problem is turned into the zero problem of an entire function．By the integral identity and residue method，the trace formula for the above eigenvalue problem of Sturm-Liouville system is obtained．

Key words：periodic boundary condition；eigenvalue；asymptotic estimate；trace formula

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Rational Approximation to $|x|$ at the Chebyshev Nodes of the Second Kind

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Abstract：Rational approximation to $|x|$ is investigated at the Chebyshev nodes of the second kind．And it is proved that the rational interpolation approximation rate is $O\left(\frac{1}{n\log n}\right)$．

Key words：rational approximation；Newman-type rational operators；the second kind of Chebyshev nodes