带有 Neumann 条件的对流扩散方程的两层紧差分格式

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摘要：对带有 Neumann 边界条件的常系数对流扩散方程，建立了一个两层有限差分格式，利用离散能量分析法给出了差分解的先验估计式，分析了差分格式解存在唯一性、收敛性以及稳定性. 并得出了差分格式在 L_∞ 范数下的收敛阶数为 $O(\tau^2+h^4)$. 通过数值算例，验证了理论分析结果是正确的.

关键词：对流扩散方程；Neumann 边界条件；隐式差分格式；先验估计；收敛性；稳定性

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0　引言

考虑带有 Neumann 边界条件的一维常系数对流扩散方程构造高阶差分格式：

$$u_t+\alpha u_x-\beta u_{xx}=f(x,t),x\in(a,b),t\in(0,T],\tag{1}$$

$$u(x,0)=\varphi(x),x\in(a,b),\tag{2}$$

$$u_x(a,t)=0;u_x(b,t)=0,t\in(0,T],\tag{3}$$

其中：α,β 为常数，且 $\beta>0$.

$$u_t+\alpha v-\beta u_{xx}=f(x,t),x\in(a,b),t\in(0,T],\tag{4}$$

$$v_t+\alpha u_{xx}-\beta v_{xx}=f_x(x,t),x\in(a,b),t\in(0,T],\tag{5}$$

$$u(x,0)=\phi(x);v(x,0)=\phi'(x),x\in(a,b),\tag{6}$$

$$v(a,t)=0;v(b,t)=0,t\in(0,T],\tag{7}$$

令 $v(x,t)=u_x(x,t)$，则方程(1)转化为方程(4)，同时对方程(1)关于 x 求一阶导数为方程(5). 方程(4)～(7)为耦合方程，与(1)～(3)式等价.

对流扩散方程是描述黏性流体运动的非线性模型方程，但要得到对流扩散方程的精确解很困难，因此有效的数值算法越来越重要，在常用的差分方法中，由于方程中扩散项的存在，在数值求解过程中经常会出现数值震荡，为此需要构造精度高、稳定性好的数值解法，紧差分格式就是这一类方法. 文献[1]给出了二维变对流系数非稳态对流扩散方程的时间方向上加权离散的一类 HOC 格式. 文献[2]给出了二维不稳定对流扩散方程的一种高阶交替方向隐格式，此方法在时间和空间上分别是二阶和四阶的. 文献[3－4]研究了对流扩散方程的特征有限差分格式，此方法能够有效地克服数值震荡. 文献[5－6]给出了关于 Neumann 边界条件热方程的高阶差分格式. 文献[7－8]通过紧差分格式及高阶 ADI 格式研究对流扩散问题. 文献[9]提出带有 Neumann 边界条件的非线性反应扩散方程的一种四阶紧算法. 文献[10]通过引入新变量，建立了一维非稳态对流扩散方程的高阶有限差分格式，利用 Von-Neumann 方法分析了差分格式的稳定性，在时间和空间上为二阶和四阶收敛. 文献[11－12]利用紧差分格式求解热方程、变系数线性抛物方程及 Cahn-Hilliard 方程. 本文参考文献[13]利用离散能量估计方法证明了差分解在最大模意义下关于时间和空间的二阶收敛性. 对带有 Dirichlet 边界条件的对流扩散方程建立高阶紧差分格式的方法很多，而处理 Neumann 边界条件

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方法比较棘手,本文针对一维对流扩散方程建立一种紧差分格式,拟从格式的相容性、截断误差、稳定性、收敛性、以及精度等方面对 Neumann 边界条件进行研究.

1　记号及引理

取正整数 m,n,记空间步长与时间步长分别为 $h=(b-a)/m,\tau=T/n$. $x_i=a+ih,(0\leqslant i\leqslant m),t_k=k\tau,(0\leqslant k\leqslant n)$. 定义 $\Omega_h=\{x_i\mid 0\leqslant i\leqslant m\},\Omega_{h\tau}=\{(x_i,t_k)\mid 0\leqslant i\leqslant m,0\leqslant k\leqslant n\}$,称 (x_i,t_k) 为节点,并设 $\{v_i^k\mid 0\leqslant i\leqslant m,0\leqslant k\leqslant n\}$ 为 $\Omega_{h\tau}$ 上的网格函数,引进下列记号:

$$v_i^{k+1/2}=\frac{1}{2}(v_i^k+v_i^{k+1}),\delta_t v_i^{k+1/2}=\frac{1}{\tau}(v_i^{k+1}-v_i^k),\delta_x v_{i-1/2}^k=\frac{1}{h}(v_i^k-v_{i-1}^k),D_{\hat{x}}v_i^k=\frac{1}{2h}(v_{i+1}^k-v_{i-1}^k),$$

$$\delta_x^2 v_i^k=\begin{cases}\dfrac{2}{h}\delta_x v_{1/2}, & i=0,\\ \dfrac{1}{h^2}(v_{i+1}^k-2v_i^k+v_{i-1}^k), & 1\leqslant i\leqslant m-1,\\ \dfrac{2}{h}\delta_x v_{m-1/2}, & i=m,\end{cases}$$

记 $v^k=\{v_i^k\mid 0\leqslant i\leqslant m\}$,则 v^k 为 Ω_h 上的 1 个网格函数,$V_h=\{v\mid v=\{v_i,0\leqslant i\leqslant m\}$ 为 Ω_h 上的网格函数,对任意的 $u,v\in V_h$, 定义平均算子、内积及范数:

$$(\mathrm{A}u)_i=\begin{cases}\dfrac{1}{6}(5u_0+u_1), & i=0,\\ \dfrac{1}{12}(u_{i-1}+10u_i+u_{i+1}), & 1\leqslant i\leqslant m-1,\\ \dfrac{1}{6}(u_{m-1}+5u_m), & i=m,\end{cases}$$

$$(u,v)=\frac{h}{2}(u_0v_0+2\sum_{i=1}^{m-1}u_iv_i+u_mv_m),(\delta_x u,\delta_x v)=h\sum_{i=0}^{m}(\delta_x u_{i+1/2})(\delta_x v_{i+1/2}),\|u\|_A^2=\|u\|^2-\frac{h^2}{12}\|\delta_x u\|^2.$$

$$\|u\|_\infty=\max_{0\leqslant i\leqslant m}|u_i|,\quad \|u\|=\sqrt{(u,u)},\quad \|\delta_x u\|=\sqrt{(\delta_x u,\delta_x u)},$$

$$(\delta_x^2 u,\delta_x^2 v)=h[\frac{1}{2}(\delta_x^2 u_0)(\delta_x^2 v_0)+\sum_{i=1}^{m}(\delta_x^2 u_i)(\delta_x^2 v_i)+\frac{1}{2}(\delta_x^2 u_m)(\delta_x^2 v_m)].$$

引理 1[14]　设 $h>0$ 和 c 为两个常数,若 $f(x)\in C^6[c-h,c+h]$,则 $\frac{1}{12}[f''(c-h)+10f''(c)+f''(c+h)]=\frac{1}{h^2}[f(c-h)-2f(c)+f(c+h)]-\frac{h^4}{240}f^{(6)}(\xi),c-h<\xi<c+h$.

引理 2[14]　设 $v\in V_h$,则 $|v|_1^2\leqslant\frac{4}{h^2}\|v\|^2$,且对任意的 $\varepsilon>0$,有 $\|v\|_\infty^2\leqslant\varepsilon|v|_1^2+(\frac{1}{\varepsilon}+\frac{1}{b-a})\cdot\|v\|^2$.

引理 3[14]　Gronwall 不等式. 设 $\{F^k,G^k\mid\geqslant 0\}$ 为非负序列,且满足 $F^{k+1}\leqslant(1+c\tau)F^k+\tau G^k,k=0,1,2,\cdots$, 其中 c 为非负数,则有 $F^k\leqslant \mathrm{e}^{ck\tau}(F^0+\tau\sum_{l=0}^{k-1}G^l),k=1,2,\cdots$

引理 4[15]　设 $u=\{u_i\mid 0\leqslant i\leqslant m\}\in V_h,v=\{v_i\mid 0\leqslant i\leqslant m\}\in V_h$,则有 $(\delta_x^2 u,v)=(u,\delta_x^2 v)=-(\delta_x u,\delta_x v),\frac{5}{12}\|u\|^2\leqslant\|\mathrm{A}u\|^2\leqslant\|u\|^2$.

引理 5[16]　① 若 $f(x)\in C^5[x_0,x_1]$, 则

$$\frac{1}{6}[5f''(x_0)+f''(x_1)]-\frac{2}{h}[\frac{f(x_1)-f(x_0)}{h}-f'(x_0)]=-\frac{h}{6}f'''(x_0)+\frac{h^3}{90}f^{(5)}(\xi),x_0<\xi<x_1.$$

② 若 $f(x)\in C^5[x_m,x_{m-1}]$, 则

$$\frac{1}{6}[5f''(x_{m-1})+f''(x_m)]-\frac{2}{h}[f'(x_m)-\frac{f(x_m)-f(x_{m-1})}{h}]=\frac{h}{6}f'''(x_m)-\frac{h^3}{90}f^{(5)}(\xi),x_{m-1}<\xi<x_m.$$

引理 6[16] 对于定义在 Ω_h 上的网格函数,有 $\frac{2}{3}\|u\|^2\leqslant\|u\|_A^2\leqslant\|u\|$.

2 差分格式的建立

设 $U=\{U_i^k\,|\,0\leqslant i\leqslant m,0\leqslant k\leqslant n\}$, $V=\{V_i^k\,|\,0\leqslant i\leqslant m,0\leqslant k\leqslant n\}$ 为定义在 $\Omega_{h\tau}$ 上的网格函数,记 $U_i^k=u(x_i,t_k)$, $V_i^k=v(x_i,t_k)$, $t_{k+1/2}=\frac{1}{2}(t_k+t_{k+1})$,现考虑在点 $(x_i,t_{k+1/2})$ 处(4)和(5)式微分方程,并利用 Taylor 展开,且 $0\leqslant i\leqslant m,0\leqslant k\leqslant n-1$, 得

$$\delta_tU_i^{k+1/2}+\alpha V_i^{k+1/2}-\frac{\beta}{2}[\frac{\partial^2u}{\partial x^2}(x_i,t_k)+\frac{\partial^2u}{\partial x^2}(x_i,t_{k+1})]=f(x_i,t_{k+1/2})+R_{ik}^{(1)},\tag{8}$$

$$\delta_tV_i^{k+1/2}+\frac{\alpha}{2}[\frac{\partial^2u}{\partial x^2}(x_i,t_k)+\frac{\partial^2u}{\partial x^2}(x_i,t_{k+1})]-\frac{\beta}{2}[\frac{\partial^2v}{\partial x^2}(x_i,t_k)+\frac{\partial^2v}{\partial x^2}(x_i,t_{k+1})]=f_x(x_i,t_{k+1/2})+R_{ik}^{(2)},\tag{9}$$

其中: $t_k\leqslant\eta_{ik}^{(1)},\eta_{ik}^{(2)},\xi_{ik}^{(0)},\xi_{ik}^{(1)},\xi_{ik}^{(2)}\leqslant t_{k+1}$;

$$R_{ik}^{(1)}=\frac{\tau^2}{24}\frac{\partial^3u}{\partial t^3}(x_i,\eta_{ik}^{(1)})+\frac{\alpha\tau^2}{8}\frac{\partial^2v}{\partial t^2}(x_i,\xi_{ik}^{(0)})-\frac{\beta\tau^2}{8}\frac{\partial^4u}{\partial x^2\partial t^2}(x_i,\xi_{ik}^{(1)});$$

$$R_{ik}^{(2)}=\frac{\tau^2}{24}\frac{\partial^3u}{\partial t^3}(x_i,\eta_{ik}^{(2)})+\frac{\alpha\tau^2}{8}\frac{\partial^2v}{\partial t^2}(x_i,\xi_{ik}^{(1)})-\frac{\beta\tau^2}{8}\frac{\partial^4u}{\partial x^2\partial t^2}(x_i,\xi_{ik}^{(2)}).$$

当 $0\leqslant i\leqslant m,0\leqslant k\leqslant n-1$ 时,用算子 A 分别作用于(8)和(9)两式得

$$\mathrm{A}\delta_tU_i^{k+1/2}+\alpha\mathrm{A}V_i^{k+1/2}-\frac{\beta}{2}[\mathrm{A}\frac{\partial^2u}{\partial x^2}(x_i,t_k)+\mathrm{A}\frac{\partial^2u}{\partial x^2}(x_i,t_{k+1})]=\mathrm{A}f(x_i,t_{k+1/2})+\mathrm{A}R_{ik}^{(1)},\tag{10}$$

$$\mathrm{A}\delta_tV_i^{k+1/2}+\frac{\alpha}{2}[\mathrm{A}\frac{\partial^2u}{\partial x^2}(x_i,t_k)+\mathrm{A}\frac{\partial^2u}{\partial x^2}(x_i,t_{k+1})]-\frac{\beta}{2}[\mathrm{A}\frac{\partial^2v}{\partial x^2}(x_i,t_k)+\mathrm{A}\frac{\partial^2v}{\partial x^2}(x_i,t_{k+1})]=\mathrm{A}f_v(x_i,t_{k+1/2})+\mathrm{A}R_{ik}^{(2)}.\tag{11}$$

当 $0\leqslant i\leqslant m-1,0\leqslant k\leqslant n-1$ 时,并由引理 1 将(10)和(11)两式转化为:

$$\mathrm{A}\delta_tU_i^{k+1/2}+\alpha\mathrm{A}V_i^{k+1/2}-\beta\delta_x^2U_i^{k+1/2}=\mathrm{A}f(x_i,t_{k+1/2})+\rho_{ik}^{(1)},\tag{12}$$

$$\mathrm{A}\delta_tV_i^{k+1/2}+\alpha\delta_x^2U_i^{k+1/2}-\beta\delta_x^2V_i^{k+1/2}=\mathrm{A}f_v(x_i,t_{k+1/2})+\rho_{ik}^{(2)},\tag{13}$$

其中: $\rho_{ik}^{(1)}=\mathrm{A}R_{ik}^{(1)}+\frac{\beta}{2}[\frac{h^4}{240}\frac{\partial^6u}{\partial x^6}(\xi_i^1,t_k)+\frac{h^4}{240}\frac{\partial^6u}{\partial x^6}(\xi_i^2,t_{k+1})]$;

$$\rho_{ik}^{(2)}=\mathrm{A}R_{ik}^{(2)}+\frac{\beta}{2}[\frac{h^4}{240}\frac{\partial^6u}{\partial x^6}(\xi_i^3,t_k)+\frac{h^4}{240}\frac{\partial^6u}{\partial x^6}(\xi_i^4,t_{k+1})]-\frac{\alpha}{2}[\frac{h^4}{240}\frac{\partial^6u}{\partial x^6}(\xi_i^1,t_k)+\frac{h^4}{240}\frac{\partial^6u}{\partial x^6}(\xi_i^2,t_{k+1})].$$

$x_i\leqslant\xi_i^1,\xi_i^2,\xi_i^3,\xi_i^4\leqslant x_{i+1}$,则存在常数 $C_0>0,C_1>0$ 使得

$|\rho_{ik}^{(1)}|\leqslant C_0(\tau^2+h^4),0\leqslant i\leqslant m-1,0\leqslant k\leqslant n-1$, $|\rho_{ik}^{(2)}|\leqslant C_1(\tau^2+h^4),0\leqslant i\leqslant m-1,0\leqslant k\leqslant n-1$.

由(10)式知,当 $i=0,0\leqslant k\leqslant n-1$ 时

$$\mathrm{A}\delta_tU_0^{k+1/2}+\alpha\mathrm{A}V_0^{k+1/2}-\frac{\beta}{2}[\mathrm{A}\frac{\partial^2u}{\partial x^2}(x_0,t_k)+\mathrm{A}\frac{\partial^2u}{\partial x^2}(x_0,t_{k+1})]=\mathrm{A}f(x_0,t_{k+1/2})+\mathrm{A}R_{0k}^{(1)}.\tag{14}$$

应用引理 5 将(14)式转化为

$$\mathrm{A}\delta_tU_0^{k+1/2}+\alpha\mathrm{A}V_0^{k+1/2}-\frac{\beta}{2}\{\frac{2}{h}[\frac{u(x_1,t_k)-u(x_0,t_k)}{h}-\frac{\partial u}{\partial x}(x_0,t_k)]-\frac{h}{6}\frac{\partial^3u}{\partial x^3}(x_0,t_k)+\frac{h^3}{90}\frac{\partial^5u}{\partial x^5}(\xi^5,t_k)+\frac{2}{h}[\frac{u(x_1,t_{k+1})-u(x_0,t_{k+1})}{h}-\frac{\partial u}{\partial x}(x_0,t_{k+1})]-\frac{h}{6}\frac{\partial^3u}{\partial x^3}(x_0,t_{k+1})+\frac{h^3}{90}\frac{\partial^5u}{\partial x^5}(\xi^6,t_{k+1})\}=\mathrm{A}f(x_0,t_{k+1/2})+\mathrm{A}R_{0k}^{(1)},0\leqslant k\leqslant n-1,x_0<\xi^5,\xi^6<x_1.\tag{15}$$

将(1)式关于 x 求导及由边界条件知 $\frac{\partial^3u}{\partial x^3}(x_0,t_k)=\frac{\alpha}{\beta^2}u_t(x_0,t_k)-\frac{\alpha}{\beta^2}f(x_0,t_k)-\frac{1}{\beta}f_x(x_0,t_k),0\leqslant k\leqslant n$, 当

$0 \leqslant k \leqslant n-1$ 时，则(15)式转化为

$$\mathrm{A}\delta_t U_0^{k+1/2} + \alpha \mathrm{A} V_0^{k+1/2} + \frac{\alpha h}{6\beta}\delta_t U_0^{k+1/2} - \frac{2\beta}{h}\delta_x U_{1/2}^{k+1/2} = \mathrm{A}f(x_0, t_{k+1/2}) + \frac{\alpha h}{6\beta} f_0^{k+1/2} + \frac{h}{6}(f_x)_0^{k+1/2} + \rho_{0k}^{(1)}, \quad (16)$$

其中：$\rho_{0k}^{(1)} = \mathrm{A}R_{0k}^{(1)} + \frac{\beta h^3}{180}\frac{\partial^5 u}{\partial x^5}(\xi^5, t_k) + \frac{\beta h^3}{180}\frac{\partial^5 u}{\partial x^5}(\xi^6, t_{k+1}) + \frac{\beta \tau^2 h}{48}\frac{\partial^3 u}{\partial x^3}(x_0, \zeta_{0k}^{(1)}) + \frac{\alpha \tau^2 h}{144}\frac{\partial^3 u}{\partial x^3}(x_0, \zeta_{0k}^{(2)})$；$t_k < \zeta_{0k}^{(1)}$；$\zeta_{0k}^{(2)} < t_{k+1}$. 类似地，当 $i = m$，$0 \leqslant k \leqslant n-1$ 时有

$$\mathrm{A}\delta_t U_m^{k+1/2} + \alpha \mathrm{A} V_m^{k+1/2} - \frac{\alpha h}{6\beta}\delta_t U_m^{k+1/2} + \frac{2\beta}{h}\delta_x U_{m-1/2}^{k+1/2} = \mathrm{A}f(x_m, t_{k+1/2}) - \frac{\alpha h}{6\beta} f_m^{k+1/2} - \frac{h}{6}(f_x)_m^{k+1/2} + \rho_{mk}^{(1)}, \quad (17)$$

其中：$\rho_{mk}^{(1)} = \mathrm{A}R_{mk}^{(1)} - \frac{\beta h^3}{180}\frac{\partial^5 u}{\partial x^5}(\xi^7, t_k) + \frac{\beta h^3}{180}\frac{\partial^5 u}{\partial x^5}(\xi^8, t_{k+1}) - \frac{\beta \tau^2 h}{48}\frac{\partial^3 u}{\partial x^3}(x_m, \zeta_{mk}^{(1)}) - \frac{\alpha \tau^2 h}{144}\frac{\partial^3 u}{\partial x^3}(x_m, \zeta_{mk}^{(2)})$；$x_{m-1} < \xi^7$；$\xi^8 < x_m$；$t_k < \zeta_{mk}^{(1)}$；$\zeta_{mk}^{(2)} < t_{k+1}$. 则存在常数 $C_3 > 0, C_4 > 0$ 使得

$$|\rho_{0k}^{(1)}| \leqslant C_3(\tau^2 + h^3), 0 \leqslant k \leqslant n-1, \quad |\rho_{mk}^{(2)}| \leqslant C_4(\tau^2 + h^3), 0 \leqslant k \leqslant n-1,$$

由初始条件(6)式，得 $v(x,0) = \varphi'(x)$，$a \leqslant x \leqslant b$，在(12) ~ (13) 式，(16) ~ (17) 式中略去小量项 $\rho_{ik}^{(1)}$，$\rho_{ik}^{(2)}$，$\rho_{0k}^{(1)}$，$\rho_{mk}^{(2)}$，并分别用 u_i^k，v_i^k 代替 U_i^k，V_i^k，可得到差分格式：

$$\mathrm{A}\delta_t u_i^{k+1/2} + \alpha \mathrm{A} v_i^{k+1/2} - \beta\delta_x^2 u_i^{k+1/2} = \mathrm{A}f(x_i, t_{k+1/2}), 0 \leqslant i \leqslant m-1, 0 \leqslant k \leqslant n-1, \quad (18)$$

$$\mathrm{A}\delta_t v_i^{k+1/2} + \alpha\delta_x^2 u_i^{k+1/2} - \beta\delta_x^2 v_i^{k+1/2} = \mathrm{A}f_v(x_i, t_{k+1/2}), 0 \leqslant i \leqslant m-1, 0 \leqslant k \leqslant n-1, \quad (19)$$

$$\mathrm{A}\delta_t u_0^{k+1/2} + \alpha \mathrm{A} v_0^{k+1/2} + \frac{\alpha h}{6\beta}\delta_t u_0^{k+1/2} - \frac{2\beta}{h}\delta_x u_{1/2}^{k+1/2} = \mathrm{A}f(x_0, t_{k+1/2}) + \frac{\alpha h}{6\beta} f_0^{k+1/2} + \frac{h}{6}(f_x)_0^{k+1/2}, 0 \leqslant k \leqslant n-1, \quad (20)$$

$$\mathrm{A}\delta_t u_m^{k+1/2} + \alpha \mathrm{A} v_m^{k+1/2} - \frac{\alpha h}{6\beta}\delta_t u_m^{k+1/2} + \frac{2\beta}{h}\delta_x u_{m-1/2}^{k+1/2} = \mathrm{A}f(x_m, t_{k+1/2}) - \frac{\alpha h}{6\beta} f_m^{k+1/2} - \frac{h}{6}(f_x)_m^{k+1/2}, 0 \leqslant k \leqslant n-1, \quad (21)$$

$$u_i^0 = \phi(x_i); v_i^0 = \phi'(x_i), 0 \leqslant i \leqslant m, \quad (22)$$

$$v_0^k = 0; v_m^k = 0, 0 \leqslant k \leqslant n, \quad (23)$$

3　差分格式解的先验估计式

定理 1　设 $\{u_i^k, v_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 是(18) ~ (23) 式的解，当取 $h \leqslant \beta/|\alpha|$ 充分小时，则有 $2\alpha^2\|u^k\|^2 + 4\beta^2\|v^k\|^2 + 2\alpha^2\beta\|\delta_x u^k\|^2 \leqslant \frac{3}{2}\mathrm{e}^{4C_6 k\tau}[2\alpha^2\|u^0\|^2 + 4\beta^2\|v^0\|^2 + 2\alpha^2\beta\|\delta_x u^0\|^2) + \tau\sum_{l=0}^{k-1}(4P^{k+1/2})]$，其中：$k = 1, 2\cdots$；$C_6 = \frac{3|\alpha|^3}{4\beta^2} + \frac{9\alpha^4}{2\beta^2} + \frac{3|\alpha|}{8} + \frac{3}{4\beta} + \frac{3}{4(b-a)} + \frac{|\alpha|}{16\beta} + 2$；$P^{k+1/2} = 2\alpha^2\{6\|\mathrm{A}f^{k+1/2}\|^2 + (\frac{1}{24} + \frac{|\alpha|}{48\beta})[(hf_0^{k+1/2})^2 + (hf_m^{k+1/2})^2] + \frac{b-a}{16}[(h(f_x)_0^{k+1/2})^2 + (h(f_x)_m^{k+1/2})^2] + \frac{\alpha^2}{2}[(h\mathrm{A}f_0^{k+1/2})^2 + (h\mathrm{A}f_m^{k+1/2})^2]\} + 2\alpha^2 h\sum_{i=1}^{m-1}(\mathrm{A}f_i^{k+1/2})^2 + \beta^2 h\sum_{i=1}^{m-1}(\mathrm{A}(f_x)_i^{k+1/2})^2$.

证明　将(18)式两边同时乘以 $h\delta_t u_i^{k+1/2}$，对 i 到 $m-1$ 求和并移项得

$$h\sum_{i=1}^{m-1}(\mathrm{A}\delta_t u_i^{k+1/2})\delta_t u_i^{k+1/2} - \beta h\sum_{i=1}^{m-1}(\delta_x^2 u_i^{k+1/2})\delta_t u_i^{k+1/2} = h\sum_{i=1}^{m-1}(\mathrm{A}f_i^{k+1/2})\delta_t u_i^{k+1/2} - \alpha h\sum_{i=1}^{m-1}(\mathrm{A}v_i^{k+1/2})\delta_t u_i^{k+1/2}, \quad (24)$$

将式(20)两边同时乘以 $(h/2)\delta_t u_0^{k+1/2}$，转化后得

$$\frac{h}{2}(\mathrm{A}\delta_t u_0^{k+1/2})\delta_t u_0^{k+1/2} - \beta(\delta_x u_{1/2}^{k+1/2})\delta_t u_0^{k+1/2} \leqslant \frac{h}{2}(\mathrm{A}f_0^{k+1/2})\delta_t u_0^{k+1/2} + \frac{|\alpha| h^2}{6\beta}(\delta_t u_0^{k+1/2})^2 + \frac{|\alpha|}{48\beta}(hf_0^{k+1/2})^2 + \frac{b-a}{48}(h(f_x)_0^{k+1/2})^2 + \frac{h}{12}(\delta_t u_0^{k+1/2})^2 - \frac{\alpha h}{2}(\mathrm{A}v_0^{k+1/2})\delta_t u_0^{k+1/2}. \quad (25)$$

将式(21)两边同时乘以 $\frac{h}{2}\delta_t u_m^{k+1/2}$，并进行转化后得

$$\frac{h}{2}(\mathrm{A}\delta_t u_m^{k+1/2})\delta_t u_m^{k+1/2} + \beta(\delta_x u_{1/2}^{k+1/2})\delta_t u_m^{k+1/2} \leqslant \frac{h}{2}(\mathrm{A}f_m^{k+1/2})\delta_t u_m^{k+1/2} + \frac{|\alpha| h^2}{6\beta}(\delta_t u_m^{k+1/2})^2 + \frac{|\alpha|}{48\beta}(hf_m^{k+1/2})^2 + \frac{b-a}{48}(h(f_x)_m^{k+1/2})^2 + \frac{h}{12}(\delta_t u_m^{k+1/2})^2 - \frac{\alpha h}{2}(\mathrm{A}v_m^{k+1/2})\delta_t u_m^{k+1/2}. \quad (26)$$

将(24)~(26)三式相加得

$$(A\delta_t u^{k+1/2},\delta_t u^{k+1/2})-\beta(\delta_x^2 u^{k+1/2},\delta_t u^{k+1/2})\leqslant 6\|Af^{k+1/2}\|^2+6\alpha^2\|v^{k+1/2}\|^2+\frac{|\alpha|}{48\beta}(hf_0^{k+1/2})^2+$$
$$\frac{b-a}{48}(h(f_x)_0^{k+1/2})^2+\frac{|\alpha|}{48\beta}(hf_m^{k+1/2})^2+\frac{b-a}{48}(h(f_x)_m^{k+1/2})^2+\frac{7}{12}\|\delta_t u^{k+1/2}\|.\tag{27}$$

将(18)式两边同时乘以 $hu_i^{k+1/2}$, 对 i 到 $m-1$ 求和并移项得

$$h\sum_{i=1}^{m-1}(A\delta_t u_i^{k+1/2})u_i^{k+1/2}-\beta h\sum_{i=1}^{m-1}(\delta_x^2 u_i^{k+1/2})u_i^{k+1/2}=h\sum_{i=1}^{m-1}(Af_i^{k+1/2})u_i^{k+1/2}-\alpha h\sum_{i=1}^{m-1}(Av_i^{k+1/2})u_i^{k+1/2}.\tag{28}$$

将(20)式两边同时乘以 $(h/2)u_0^{k+1/2}$, 转化得

$$\begin{aligned}&\frac{h}{2}(A\delta_t u_0^{k+1/2})u_0^{k+1/2}+\frac{\alpha h}{2}(Av_0^{k+1/2})u_0^{k+1/2}-\beta(\delta_x u_{1/2}^{k+1/2})u_0^{k+1/2}=\frac{h}{2}(Af_0^{k+1/2})u_0^{k+1/2}-\\&\frac{\alpha h^2}{12\beta}(\delta_t u_0^{k+1/2})u_0^{k+1/2}+\frac{\alpha h^2}{12\beta}(f_0^{k+1/2})u_0^{k+1/2}+\frac{h^2}{12}(f_x)_0^{k+1/2}u_0^{k+1/2},0\leqslant k\leqslant n-1.\end{aligned}\tag{29}$$

将(21)式两边同时乘以 $(h/2)u_m^{k+1/2}$, 转化后得

$$\frac{h}{2}(A\delta_t u_m^{k+1/2})u_m^{k+1/2}+\frac{\alpha h}{2}(Av_m^{k+1/2})u_m^{k+1/2}+\beta(\delta_x u_{m-1/2}^{k+1/2})u_m^{k+1/2}=\frac{h}{2}(Af_m^{k+1/2})u_m^{k+1/2}+$$
$$\frac{\alpha h^2}{12\beta}(\delta_t u_m^{k+1/2})u_m^{k+1/2}-\frac{\alpha h^2}{12\beta}(f_m^{k+1/2})u_m^{k+1/2}-\frac{h^2}{12}(f_x)_m^{k+1/2}u_m^{k+1/2},0\leqslant k\leqslant n-1.\tag{30}$$

将(28)~(30)三式相加得 $D_1+D_2=D_3+D_4+D_5$, 其中: $D_1=(A\delta_t u^{k+1/2},u^{k+1/2})=\frac{1}{2\tau}(\|u^{k+1}\|_A^2-\|u^k\|_A^2)$; $D_2=-\beta(\delta_x^2 u^{k+1/2},u^{k+1/2})=\beta\|\delta_x u^{k+1/2}\|^2$; $D_3=-\alpha(Av^{k+1/2},u^{k+1/2})\leqslant|\alpha|\|v^{k+1/2}\|^2+\frac{|\alpha|}{4}\cdot\|u^{k+1/2}\|^2$; $D_4=\frac{h^2}{12}[-\frac{\alpha}{\beta}(\delta_t u_0^{k+1/2})+\frac{\alpha}{\beta}(f_0^{k+1/2})-(f_x)_0^{k+1/2}]u_0^{k+1/2}+\frac{h^2}{12}[\frac{\alpha}{\beta}(\delta_t u_m^{k+1/2})-\frac{\alpha}{\beta}(f_m^{k+1/2})-(f_x)_m^{k+1/2}]u_m^{k+1/2}\leqslant(\frac{|\alpha|}{48\beta}+\frac{1}{24})(\|u^{k+1}\|^2+\|u^k\|^2)+(\frac{|\alpha|}{48\beta}+\frac{1}{24})[(hf_0^{k+1/2})^2+(hf_m^{k+1/2})^2]+\frac{b-a}{16}[(h(f_x)_0^{k+1/2})^2+(h(f_x)_m^{k+1/2})^2]+6\alpha^2\|v^{k+1/2}\|^2+6\|Af^{k+1/2}\|^2-\beta(\delta_x u^{k+1/2},\delta_x\delta_t u^{k+1/2})$; $D_5=(Af^{k+1/2},u^{k+1/2})\leqslant\frac{1}{2}[\beta\|\delta_x u^{k+1/2}\|^2+(\frac{1}{\beta}+\frac{1}{b-a})\|u^{k+1/2}\|^2+\frac{1}{4}\|u^{k+1/2}\|^2+\frac{1}{4}[(hAf_0^{k+1/2})^2+(hAf_m^{k+1/2})^2]+h\sum_{i=1}^{m-1}(Af_i^{k+1/2})^2$.

将 D_1,D_2,D_3,D_4,D_5 代入 $D_1+D_2=D_3+D_4+D_5$ 得

$$\frac{1}{2\tau}(\|u^{k+1}\|_A^2-\|u^k\|_A^2)+\frac{\beta}{2}\|\delta_x u^{k+1/2}\|^2\leqslant(|\alpha|+6\alpha^2)\|v^{k+1/2}\|^2+[\frac{|\alpha|}{8}+\frac{1}{4\beta}+\frac{1}{4(b-a)}+\frac{|\alpha|}{48\beta}+\frac{1}{6}]\cdot$$
$$(\|u^{k+1}\|^2+\|u^k\|^2)+\frac{b-a}{16}[(h(f_x)_0^{k+1/2})^2+(h(f_x)_m^{k+1/2})^2]+\frac{1}{4}[(hAf_0^{k+1/2})^2+(hAf_m^{k+1/2})^2]+$$
$$h\sum_{i=1}^{m-1}(Af_i^{k+1/2})^2-\beta(\delta_x u^{k+1/2},\delta_x\delta_t u^{k+1/2})+6\|Af^{k+1/2}\|^2+(\frac{|\alpha|}{48\beta}+\frac{1}{24})[(hf_0^{k+1/2})^2+(hf_m^{k+1/2})^2].\tag{31}$$

将(19)式两边同时乘以 $hv_i^{k+1/2}$, 对 i 到 $m-1$ 求和并移项得

$$h\sum_{i=1}^{m-1}(A\delta_t v_i^{k+1/2})v_i^{k+1/2}-\beta h\sum_{i=1}^{m-1}(\delta_x^2 v_i^{k+1/2})v_i^{k+1/2}=h\sum_{i=1}^{m-1}(A(f_x)_i^{k+1/2})v_i^{k+1/2}-\alpha h\sum_{i=1}^{m-1}(\delta_x^2 u_i^{k+1/2})v_i^{k+1/2}.\tag{32}$$

(32)式左边第1项、第2项分别为:

$$h\sum_{i=1}^{m-1}(A\delta_t v_i^{k+1/2})v_i^{k+1/2}=(A\delta_t v^{k+1/2},v^{k+1/2})=\frac{1}{2\tau}(\|v^{k+1}\|_A^2-\|v^k\|_A^2),$$
$$-\beta h\sum_{i=1}^{m-1}(\delta_x^2 v_i^{k+1/2})v_i^{k+1/2}=-\beta(\delta_x^2 v^{k+1/2},v^{k+1/2})=\beta\|\delta_x v^{k+1/2}\|^2.$$

(32)式右边第1项、第2项分别为:

$$h\sum_{i=1}^{m-1}(\mathrm{A}(f_x)_i^{k+1/2})v_i^{k+1/2}\leqslant\|v^{k+1/2}\|^2+\frac{1}{4}h\sum_{i=1}^{m-1}(\mathrm{A}(f_x)_i^{k+1/2})v_i^{k+1/2},$$

$$-\alpha h\sum_{i=1}^{m-1}(\delta_x^2u_i^{k+1/2})v_i^{k+1/2}=-\alpha(\delta_x^2u^{k+1/2},v^{k+1/2})\leqslant\varepsilon|\alpha|\|\delta_xv^{k+1/2}\|^2+\frac{|\alpha|}{4\varepsilon}\|\delta_xu^{k+1/2}\|^2,$$

其中 $\varepsilon=-\beta/|\alpha|$，将上述 4 项代入(32)式得

$$\frac{1}{2\tau}(\|v^{k+1}\|_{\mathrm{A}}^2-\|v^k\|_{\mathrm{A}}^2)\leqslant\frac{\alpha^2}{4\beta}\|\delta_xu^{k+1/2}\|^2+\|v^{k+1/2}\|^2+\frac{1}{4}h\sum_{i=1}^{m-1}(\mathrm{A}(f_x)_i^{k+1/2})v_i^{k+1/2}.\tag{33}$$

记 $C_5=\frac{|\alpha|}{8}+\frac{1}{4\beta}+\frac{1}{4(b-a)}+\frac{|\alpha|}{48\beta}+\frac{1}{6}$，将(33)式乘以 $4\beta^2$ 与(31)式乘以 $2\alpha^2$ 相加后，再在式子两边同时乘以 2τ 整理后，利用引理 6 得

$$\begin{aligned}&(2\alpha^2\|u^{k+1}\|_{\mathrm{A}}^2+4\beta^2\|v^{k+1}\|_{\mathrm{A}}^2+2\alpha^2\beta\|\delta_xu^{k+1}\|^2)-(2\alpha^2\|u^k\|_{\mathrm{A}}^2+4\beta^2\|v^k\|_{\mathrm{A}}^2+2\alpha^2\beta\|\delta_xu^k\|^2)\leqslant\\&\frac{3\tau}{2}\Big(\frac{|\alpha|^3}{2\beta}+1+\frac{3\alpha^4}{\beta^2}\Big)[4\beta^2(\|v^{k+1}\|_{\mathrm{A}}^2+\|v^k\|_{\mathrm{A}}^2)+2\alpha^2(\|u^{k+1}\|_{\mathrm{A}}^2+\|u^k\|_{\mathrm{A}}^2)]+\\&3\tau C_5[2\alpha^2(\|u^{k+1}\|_{\mathrm{A}}^2+\|u^k\|_{\mathrm{A}}^2)+4\beta^2(\|v^{k+1}\|_{\mathrm{A}}^2+\|v^k\|_{\mathrm{A}}^2)]+2\tau P^{k+1/2},\end{aligned}\tag{34}$$

其中 $P^{k+1/2}=2\alpha^2\{6\|\mathrm{A}f^{k+1/2}\|^2+\Big(\frac{|\alpha|}{48\beta}+\frac{1}{24}\Big)[(hf_0^{k+1/2})^2+(hf_m^{k+1/2})^2]+\frac{b-a}{16}[(h(f_x)_0^{k+1/2})^2+(h(f_x)_m^{k+1/2})^2\}+\frac{\alpha^2}{2}[(h\mathrm{A}f_0^{k+1/2})^2+(h\mathrm{A}f_m^{k+1/2})^2]+2\alpha^2h\sum_{i=1}^{m-1}(\mathrm{A}f_i^{k+1/2})^2+\beta^2(\mathrm{A}(f_x)_i^{k+1/2})^2$.

记 $Q^k=2\alpha^2\|u^k\|_{\mathrm{A}}^2+4\beta^2\|v^k\|_{\mathrm{A}}^2+2\alpha^2\beta\|\delta_xu^k\|^2$，$C_6=\frac{3|\alpha|^3}{4\beta^2}+\frac{5\alpha^4}{2\beta^2}+\frac{3}{2}+3C_5$，则(34)式为 $Q^{k+1}-Q^k\leqslant\tau C_6(Q^{k+1}+Q^k)+2\tau P^{k+1/2}$，当 $\tau\leqslant\frac{1}{2C_6}$ 时，$Q^{k+1}\leqslant(1+4\tau C_6)Q^k+4\tau P^{k+1/2}$，由 Gronwall 不等式得 $Q^k\leqslant\mathrm{e}^{4C_6k\tau}[Q^0+\tau\sum_{l=0}^{k-1}(4P^{l+1/2})]$，$k=1,2,3,\cdots$

定理 2 得证.

4 差分格式的唯一性、收敛性和稳定性

1）唯一性

定理 2 差分格式(18)～(23)式是唯一可解.

证明 差分格式(18)～(23)式是线性的，考虑其对应的齐次方程组，由定理 1 知 $2\alpha^2\|u^k\|^2+4\beta^2\|v^k\|^2+2\alpha^2\beta\|\delta_xu^k\|^2\leqslant\frac{3}{2}\mathrm{e}^{4C_6k\tau}[2\alpha^2\|u^0\|^2+4\beta^2\|v^0\|^2+2\alpha^2\beta\|\delta_xu^0\|^2)+\tau\sum_{l=0}^{k-1}(4P^{l+1/2})]=0$. 易知 $u_i^k=0,v_i^k=0,0\leqslant i\leqslant m,0\leqslant k\leqslant n$，则差分格式(22)～(27)式是唯一可解的，定理得证.

2）收敛性

定理 3 设 $u(x_i,t_k),v(x_i,t_k)$ 是(3)～(8)式的解，$\{u_i^k,v_i^k|0\leqslant i\leqslant m,0\leqslant k\leqslant n\}$ 是差分格式(18)～(23)式的解，记 $\mathrm{e}_i^k=u(x_i,t_k)-u_i^k$，$\hat{\mathrm{e}}_i^k=v(x_i,t_k)-v_i^k$，$0\leqslant i\leqslant m,0\leqslant k\leqslant n$，则当取 h,τ 充分小时有 $2\alpha^2\|\mathrm{e}^k\|^2+4\beta^2\|\hat{\mathrm{e}}^k\|^2+2\alpha^2\beta\|\delta_x\mathrm{e}^k\|^2\leqslant C(\tau^2+h^4)^2$，$k=0,1,2,\cdots$

证明 将(3)～(7)式与(18)～(23)式分别相减，得到误差方程组：

$$\mathrm{A}\delta_t\mathrm{e}_i^{k+1/2}+\alpha\mathrm{A}\hat{\mathrm{e}}_i^{k+1/2}-\beta\delta_x^2\mathrm{e}_i^{k+1/2}=\rho_{ik}^{(1)},0\leqslant i\leqslant m-1,0\leqslant k\leqslant n-1,$$

$$\mathrm{A}\delta_t\hat{\mathrm{e}}_i^{k+1/2}+\alpha\delta_x^2\mathrm{e}_i^{k+1/2}-\beta\delta_x^2\hat{\mathrm{e}}_i^{k+1/2}=\rho_{ik}^{(2)},0\leqslant i\leqslant m-1,0\leqslant k\leqslant n-1,$$

$$\mathrm{A}\delta_t\mathrm{e}_0^{k+1/2}+\alpha\mathrm{A}\hat{\mathrm{e}}_0^{k+1/2}+\frac{\alpha h}{6\beta}\delta_t\mathrm{e}_0^{k+1/2}-\frac{2\beta}{h}\delta_x\mathrm{e}_{1/2}^{k+1/2}=\rho_{0k}^{(1)},0\leqslant k\leqslant n-1,$$

$$\mathrm{A}\delta_t\mathrm{e}_m^{k+1/2}+\alpha\mathrm{A}\hat{\mathrm{e}}_m^{k+1/2}-\frac{\alpha h}{6\beta}\delta_t\mathrm{e}_m^{k+1/2}+\frac{2\beta}{h}\delta_x\mathrm{e}_{m-1/2}^{k+1/2}=0,0\leqslant k\leqslant n-1,$$

$$\mathrm{e}_i^0=0,\hat{\mathrm{e}}_i^0=0,0\leqslant i\leqslant m,\ \hat{\mathrm{e}}_0^k=0,\hat{\mathrm{e}}_m^k=0,0\leqslant k\leqslant n.$$

由定理 2 知，定理 3 成立.

3）稳定性

类似讨论差分格式的收敛性，可以得到差分格式(18)~(23)式关于初值的稳定性.

定理4 设 $\{u_i^k, v_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 是差分格式(18)~(23)式的解，设 $\{\hat{u}_i^k, \hat{v}_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 也是差分格式(22)~(27)式的解，记 $\hat{\varepsilon}_i^k = \hat{u}_i^k - u_i^k, \hat{\xi}_i^k = \hat{v}_i^k - v_i^k, \hat{g}_i^k = \hat{f}_i^k - f_i^k, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n$，则当取 $h \leqslant \frac{\beta}{|\alpha|}$，$\tau$ 充分小时，有 $2\alpha^2 \|\hat{\varepsilon}^k\|^2 + 4\beta^2 \|\hat{\xi}^k\|^2 + 2\alpha^2\beta \|\delta_x \hat{\varepsilon}^k\|^2 \leqslant \frac{3}{2} e^{4C_6 k\tau} [2\alpha^2 \|\hat{\varepsilon}^0\|^2 + 4\beta^2 \|\hat{\xi}^0\|^2 + 2\alpha^2\beta \|\delta_x \hat{\varepsilon}^0\|^2) + \tau \sum_{l=0}^{k-1} (4P^{l+1/2})]$，其中 $P^{k+1/2} = 2\alpha^2 \{6 \|A\hat{f}^{k+1/2}\|^2 + (\frac{|\alpha|}{48\beta} + \frac{1}{24})[(h\hat{f}_0^{k+1/2})^2 + (h\hat{f}_m^{k+1/2})^2] + \frac{b-a}{16}[(h(\hat{f}_x)_0^{k+1/2})^2 + (h(\hat{f}_x)_m^{k+1/2})^2]\} + \frac{\alpha^2}{2}[(hA\hat{f}_0^{k+1/2})^2 + (hA\hat{f}_m^{k+1/2})^2] + 2\alpha^2 h \sum_{i=1}^{m-1} (A\hat{f}_i^{k+1/2})^2 + \beta^2 (A(\hat{f}_x)_i^{k+1/2})^2$. 直接应用定理2，即可得到定理4.

5 数值试验

设 $E_i^k = u(x_i, t_k) - u_i^k$，则 $\|E\|_\infty = \max_{0 \leqslant k \leqslant n} \|u(x_i, t_k) - u_i^k\|_\infty$. 利用差分格式计算实例，表1给出了不同步长时的最大误差及误差比，从计算结果可以看出，建立的差分格式在无穷范数下的收敛阶 $O(\tau^2 + h^4)$，也更充分说明数值试验的解与理论分析结果吻合.

例
$$\begin{cases} u_t + 0.1u_x - 2u_{xx} = -0.01e^{2t}\sin x + 0.4e^{2t}\cos x, & 0 \leqslant x \leqslant \pi, 0 < t < 1, \\ u(x,0) = 0.1\cos x, & 0 \leqslant x \leqslant \pi, \\ u_x(0,t) = u_x(\pi,t) = 0, & 0 \leqslant t \leqslant 1. \end{cases}$$

该问题的精确解为 $u(x,t) = 0.1e^{2t}\cos x$.

表1 不同步长下的误差和收敛阶数

Tab.1 Errors and convergence rate under different steps

$\frac{(h,\tau)}{(x,t)}$	$\|E\|_\infty(h,\tau)$	$\frac{\|E(h,4\tau)\|_\infty}{\|E_\infty(h/2,\tau)\|_\infty}$	$\frac{(h,\tau)}{(x,t)}$	$\|E\|_\infty(h,\tau)$	$\frac{\|E(h,4\tau)\|_\infty}{\|E_\infty(h/2,\tau)\|_\infty}$
$(\frac{\pi}{100},\frac{1}{10})$	0.002 449	—	$(\frac{\pi}{10},\frac{1}{100\ 000})$	$2.001\ 262 \times 10^{-5}$	—
$(\frac{\pi}{100},\frac{1}{20})$	0.000 613	3.992 6	$(\frac{\pi}{20},\frac{1}{100\ 000})$	$1.251\ 774 \times 10^{-6}$	15.987 4
$(\frac{\pi}{100},\frac{1}{40})$	0.000 153	3.998 2	$(\frac{\pi}{30},\frac{1}{100\ 000})$	$7.823\ 052 \times 10^{-8}$	16.001 1
$(\frac{\pi}{100},\frac{1}{60})$	0.000 038	3.999 6	$(\frac{\pi}{40},\frac{1}{100\ 000})$	$4.865\ 686 \times 10^{-9}$	16.078 0

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A Two-level Finite Difference Scheme for Convection Diffusion Equations with Neumann Boundary Conditions

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Abstract: A two-level finite difference scheme was established for the initial-boundary problem of convection diffusion equation with Neumann boundary condition. A prior estimate could be got by using the discrete energy analysis. Simultaneously, the uniqueness, convergence, and stability of difference scheme were analyzed. The convergence order of the difference scheme in maximum norm was $O(\tau^2 + h^4)$. And numerical results were conducted to illustrate the theoretical results of the presented scheme.

Key words: convection diffusion equation; Neumann boundary value; difference scheme; prior estimate; convergence; stability

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