带 Neumann 边界条件的抛物型方程的样条差分方法

刘 蕤[1,2], 高锐敏[3]

(1. 郑州幼儿师范高等专科学校 理科教学部 河南 郑州 450000; 2. 北京师范大学 教育管理学院 北京 100875; 3. 河南牧业经济学院 基础部 河南 郑州 450044)

摘要: 基于四次样条函数和广义梯形公式,针对抛物型方程的 Neumann 边值问题,构造了一族含参数 θ($\theta \in [0,1]$)的隐式差分格式,该格式在时间方向的精度为二阶,在空间方向的精度为四阶,当 $\theta = 1/3$ 时,该差分格式在时间方向的精度可提高到三阶. 数值实验表明方法是非常有效的.

关键词: 抛物型方程; 四次样条函数; 差分格式; Neumann 边值问题

中图分类号: O 241.83 文献标志码: A 文章编号: 1671-6841(2013)03-0037-04

DOI: 10.3969/j.issn/1671-6841.2013.03.009

0 引言

考虑带 Neumann 边界条件的一维抛物型方程

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(x,t), a > 0, 0 < x < L, t \geqslant 0, \tag{1}$$

$$u(x,0) = \varphi(x), 0 \leqslant x \leqslant L, \tag{2}$$

$$u_x(0,t) = \alpha(t); u_x(L,t) = \beta(t), t \geqslant 0. \tag{3}$$

对于一维抛物方程的 Dirichlet 边值问题,已有一些有限差分格式[1-5]. 而对于一维抛物方程的 Neumann 边值问题,针对导数边界条件的离散,传统的差分方法只具有一阶或二阶精度. 最近,文[6]等给出了三次 B 样条方法,该方法在内部节点的截断误差为 $O(k+h^2)$,而在边界点的截断误差为 $O(k+h^3)$. 文[7]提出了无条件稳定的紧致差分格式,该差分格式在所有节点的截断误差都达到了 $O(k^2+h^4)$.

基于四次多项式样条函数和广义梯形公式[8],本文给出了一类求解抛物型方程 Neumann 边值问题的含参数 θ 的隐式差分格式,其中 $\theta \in [0,1]$. 当 $\theta \neq 1/3$ 时,这些差分格式在所有节点的截断误差为 $O(k^2+h^4)$; 当 $\theta = 1/3$ 时,该差分格式的局部截断误差为 $O(k^3+h^4)$. 稳定性分析表明这类差分格式是无条件稳定的.

1 四次样条函数及插值误差估计

设有一均匀剖分 $\Pi: 0 = x_0 < x_1 < x_2 < \cdots < x_n = L$,其中,$x_i = ih, h = L/n$.

如果函数 $s(x)$ 满足条件:

(a) $s(x)$ 在每一个小区间是一个四次多项式函数;

(b) $s(x) \in C^3[0,1]$,即 $s(x)$ 于 $[0,1]$ 上具有连续的一阶、二阶和三阶导数,则称为关于剖分 Π 的一个四次样条函数,且该样条函数空间的维数为 $n+4$.

由四次样条函数 $s(x)$ 的定义可知,在每一个区间 $[x_{i-1}, x_i]$ $(i = 1,2,\cdots,n)$ 上,$s(x)$ 可表示成

$$s(x) = s(x_{i-1}) + hs'(x_{i-1})t + h^2 s''(x_{i-1})\frac{t^2}{12}(6-t^2) + h^2 s''(x_i)\frac{t^4}{12} + h^3 s'''(x_{i-1})\frac{t^3}{12}(2-t). \tag{4}$$

收稿日期: 2013-03-06

基金项目: 河南省基础与前沿技术研究计划项目,编号 132300410381.

作者简介: 刘蕤(1972-),女,高级讲师,主要从事偏微分方程数值方法研究,E-mail: liurui72888@163.com.

基于(4)式,进一步可得[9]

$$\frac{1}{12}s''(x_{i+1})+\frac{5}{6}s''(x_i)+\frac{1}{12}s''(x_{i-1})=\frac{1}{h^2}[s(x_{i+1})-2s(x_i)+s(x_{i-1})],\ i=1,2,\cdots,n-1. \tag{5}$$

定理 1[10-11] 假设 $g(x)$ 在区间 [0,1] 上足够光滑,四次样条函数 $s(x)$ 为 $g(x)$ 的插值函数,且满足插值条件

$$g(x_i)=s(x_i),i=0,1,\cdots,n,\ g'(x_0)=s'(x_0),g'(x_n)=s'(x_n),g'''(x_n)=s'''(x_n),$$

则有

$$s''(x_i)=g''(x_i)+O(h^4),i=0,1,\cdots,n,$$

$$g'(x_0)=\frac{g(x_1)-g(x_0)}{h}-\frac{5}{12}hs''(x_0)-\frac{1}{12}hs''(x_1)-\frac{1}{12}h^2g'''(x_0)+O(h^4),$$

$$g'(x_n)=\frac{g(x_n)-g(x_{n-1})}{h}+\frac{5}{12}hs''(x_n)+\frac{1}{12}hs''(x_{n-1})-\frac{1}{12}h^2g'''(x_n)+O(h^4).$$

2 差分格式的建立

首先对区域进行等距剖分,令 $h=L/n$ 和 k 分别为空间和时间的步长. 设(1)式的精确解为 $u(x,t)$,u_i^j 表示 $u(x_i,t_j)$,其中,$x_i=ih,i=0,1,\cdots,n;t_j=jk,j=0,1,\cdots$,对于任意的 $t>0$,设四次样条函数 $s(x,t)$ 为 $u(x,t)$ 的插值函数,且满足插值条件

$$u(x_i,t)=s(x_i,t),i=0,1,\cdots,n,$$

$$\frac{\partial u}{\partial x}(x_0,t)=\frac{\partial s}{\partial x}(x_0,t),\frac{\partial u}{\partial x}(x_n,t)=\frac{\partial s}{\partial x}(x_n,t),\frac{\partial^3 u}{\partial x^3}(x_n,t)=\frac{\partial^3 s}{\partial x^3}(x_n,t).$$

则由定理 1 有

$$\frac{\partial^2 u(x_i,t)}{\partial x^2}=\frac{\partial^2 s(x_i,t)}{\partial x^2}+O(h^4),i=0,1,\cdots,n, \tag{6}$$

$$\frac{\partial u(x_0,t)}{\partial x}=\frac{u(x_1,t)-u(x_0,t)}{h}-\frac{5h}{12}\frac{\partial^2 s(x_0,t)}{\partial x^2}-\frac{h}{12}\frac{\partial^2 s(x_1,t)}{\partial x^2}-\frac{h^2}{12}\frac{\partial^3 u(x_0,t)}{\partial x^3}+O(h^4), \tag{7}$$

$$\frac{\partial u(x_n,t)}{\partial x}=\frac{u(x_n,t)-u(x_{n-1},t)}{h}+\frac{5h}{12}\frac{\partial^2 s(x_n,t)}{\partial x^2}+\frac{h}{12}\frac{\partial^2 s(x_{n-1},t)}{\partial x^2}-\frac{h^2}{12}\frac{\partial^3 u(x_n,t)}{\partial x^3}+O(h^4).$$

记 $u_i(t)=u(x_i,t),s_i(t)=s(x_i,t),f_i(t)=f(x_i,t)$,则方程(1)可写成

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t}=a\frac{\partial^2 u}{\partial x^2}(x_i,t)+f_i(t),i=0,1,\cdots,n. \tag{8}$$

结合(6)式和(8)式,有

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t}=a\frac{\partial^2 s}{\partial x^2}(x_i,t)+f_i(t)+O(h^4),i=0,1,\cdots,n. \tag{9}$$

忽略(9)式中误差项,并将其写成

$$\frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t}=a\boldsymbol{M}(t)+\boldsymbol{f}(t), \tag{10}$$

其中,

$$\boldsymbol{U}(t)=[u_0(t),u_1(t),\cdots,u_{n-1}(t),u_n(t)]^{\mathrm{T}},\boldsymbol{f}(t)=[f_0(t),f_1(t),\cdots,f_{n-1}(t),f_n(t)]^{\mathrm{T}},$$

$$\boldsymbol{M}(t)=\left[\frac{\partial^2 s}{\partial x^2}(x_0,t),\frac{\partial^2 s}{\partial x^2}(x_1,t),\cdots,\frac{\partial^2 s}{\partial x^2}(x_{n-1},t),\frac{\partial^2 s}{\partial x^2}(x_n,t)\right]^{\mathrm{T}}.$$

由(5)式和(6)式有

$$\frac{\partial^2 s}{\partial x^2}(x_{n-1})+10\frac{\partial^2 s}{\partial x^2}(x_i,t)+\frac{\partial^2 s}{\partial x^2}(x_{i+1},t)=\frac{12}{h^2}(s(x_{i-1},t)-2s(x_i,t)+s(x_{i+1},t))$$

$$=\frac{12}{h^2}(u_{i-1}(t)-2u_i(t)+u_{i+1}(t)). \tag{11}$$

由(1)式和(3)式可得

$$\frac{\partial^3 u(x_0,t)}{\partial x^3}=\frac{\partial}{\partial x}\left(\frac{\partial u(x,t)}{\partial t}-f(x,t)\right)\Big|(x_0,t)=\alpha'(t)-\frac{\partial f(x_0,t)}{\partial x}. \tag{12}$$

将(12)代入(7)式,并忽略其误差项,得

$$\frac{5}{12}h\frac{\partial^2 s(x_0,t)}{\partial x^2}+\frac{1}{12}h\frac{\partial^2 s(x_1,t)}{\partial x^2}=\frac{1}{h}u(x_1,t)-\frac{1}{h}u(x_0,t)-\frac{1}{12}h^2\alpha'(t)-\alpha(t)+\frac{1}{12}h^2\frac{\partial f(x_0,t)}{\partial x}. \tag{13}$$

同理可得

$$\frac{5}{12}h\frac{\partial^2 s(x_n,t)}{\partial x^2}+\frac{1}{12}h\frac{\partial^2 s(x_{n-1},t)}{\partial x^2}=\frac{1}{h}u(x_{n-1},t)-\frac{1}{h}u(x_n,t)+\beta(t)+\frac{1}{12}h^2\beta'(t)-\frac{1}{12}h^2\frac{\partial f(x_n,t)}{\partial x}. \tag{14}$$

结合(11)、(13)和(14)式,并将其写成矩阵形式

$$\boldsymbol{M}(t)=\boldsymbol{A}^{-1}\boldsymbol{C}\boldsymbol{U}(t)+\boldsymbol{A}^{-1}\boldsymbol{q}(t), \tag{15}$$

其中

$$\boldsymbol{A}=\begin{bmatrix}5 & 1 & & & \\ 1 & 10 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 10 & 1\\ & & & 1 & 5\end{bmatrix}_{\overline{n+1}\times\overline{n+1}},\quad \boldsymbol{C}=\frac{12}{h^2}\begin{bmatrix}-1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1\\ & & & 1 & -1\end{bmatrix}_{\overline{n+1}\times\overline{n+1}},$$

$$\boldsymbol{q}(t)=\left[-h\alpha'(t)-\frac{12}{h}\alpha(t)+h\frac{\partial f(x_0,t)}{\partial x},\underbrace{0,\cdots,0}_{n-1},h\beta'(t)+\frac{12}{h}\beta(t)-h\frac{\partial f(x_n,t)}{\partial x}\right]^{\mathrm T}.$$

将(15)代入(10)得

$$\begin{cases}\dfrac{\mathrm d\boldsymbol{U}(t)}{\mathrm dt}=\boldsymbol{D}\boldsymbol{U}(t)+a\boldsymbol{A}^{-1}\boldsymbol{q}(t)+\boldsymbol{f}(t)\\ \boldsymbol{U}(0)=\boldsymbol{U}_0\end{cases},$$

其中 $\boldsymbol{D}=a\boldsymbol{A}^{-1}\boldsymbol{C}$.

由文[8]的广义梯形公式可得到差分格式

$$\left(\boldsymbol{I}-\frac{1}{2}(1+\theta)k\boldsymbol{D}+\frac{1}{2}\theta k^2\boldsymbol{D}^2\right)\boldsymbol{U}_{j+1}=\left(\boldsymbol{I}+\frac{1}{2}(1-\theta)k\boldsymbol{D}\right)\boldsymbol{U}_j+\frac{k}{2}\boldsymbol{p}_j+\frac{k}{2}(\boldsymbol{I}-\theta k\boldsymbol{D})\boldsymbol{p}_{j+1}, \tag{16}$$

其中,$\boldsymbol{U}_j=\boldsymbol{U}(t_j)$,$\boldsymbol{p}(t_j)=a\boldsymbol{A}^{-1}\boldsymbol{q}(t_j)+\boldsymbol{f}(t_j)$. 由文[8]可知,当 $\theta\neq 1/3$ 时,格式(16)的截断误差为 $O(k^2+h^4)$;当 $\theta=1/3$ 时,格式(16)截断误差为 $O(k^3+h^4)$.

与文[8]类似,可证明差分格式(16)是无条件稳定的.

3　数值试验与讨论

考虑带 Neumann 边界条件的抛物线方程,在问题(1)~(3)中,取精确解为

$$u(x,t)=\left(x-\frac{1}{3}\right)^6\mathrm e^t,a=1.$$

而 $f(x,t)$,$\varphi(x)$,$\alpha(t)$,$\beta(t)$ 均可由 $u(x,t)$ 获得.

取 $\theta=1/3$,利用(16)计算空间方向的收敛阶,并与文[7]中数值结果进行比较,见表1,表2. 设 $\{U_i^j|0<i<n,0<j<N\}$ 和 $\{u_i^j|0<i<n,0<j<N\}$ 分别是方程的数值解和精确解,在表1中,定义

$$E_2(h,k)=\sqrt{h\sum_{i=1}^{n-1}(U_i^N-u_i^N)^2},\ E_\infty(h,k)=\max_{1\leqslant i\leqslant n-1}|U_i^N-u_i^N|.$$

表1,2给出了$k = 10^{-4}$时的数值结果和收敛阶,由表1,2的数值结果可知,文[7]中的(1.4)~(1.7)的收敛阶为3,而本文(16)的收敛阶为4,且数值结果明显比文[7]中的(1.4)~(1.7)的数值结果更精确,其原因在于本文(16)的局部截断误差为$O(k^3 + h^4)$,而文[7]中的(1.4)~(1.7)的局部截断误差为$O(k^2 + h^3)$.

表1 文[7]中(1.4)~(1.7)的数值结果和收敛阶

Tab. 1 Numerical convergence rate of scheme (1.4)~(1.7) in reference [7] $k = 10^{-4}$

h	$E_2(h,k)$	$\frac{E_2(h,k)}{E_2(h/2,k)}$	$\log_2\left(\frac{E_2(h,k)}{E_2(h/2,k)}\right)$	$E_\infty(h,k)$	$\frac{E_\infty(h,k)}{E_\infty(h/2,k)}$	$\log_2\left(\frac{E_\infty(h,k)}{E_\infty(h/2,k)}\right)$
1/10	0.806 1E-1	6.250	2.644	0.107 9E-0	6.259	2.646
1/20	0.129 0E-1	7.112	2.830	0.172 3E-1	7.112	2.830
1/40	0.181 4E-2	7.554	2.917	0.242 3E-2	7.776	2.959
1/80	0.240 1E-3	7.776	2.959	0.320 8E-3	7.776	2.959
1/160	0.308 8E-4	7.888	2.980	0.412 6E-4	7.888	2.980
1/320	0.391 4E-5	*	*	0.523 1E-5	*	*

表2 本文(16)的数值结果和收敛阶

Tab. 2 Numerical convergence rate of the scheme (16) $k = 10^{-4}$

h	$E_2(h,k)$	$\frac{E_2(h,k)}{E_2(h/2,k)}$	$\log_2\left(\frac{E_2(h,k)}{E_2(h/2,k)}\right)$	$E_\infty(h,k)$	$\frac{E_\infty(h,k)}{E_\infty(h/2,k)}$	$\log_2\left(\frac{E_\infty(h,k)}{E_\infty(h/2,k)}\right)$
1/10	0.202 5E-5	16.811	4.071	0.277 5E-5	14.328	3.841
1/20	0.120 5E-6	16.427	4.038	0.193 6E-6	15.159	3.922
1/40	0.733 4E-8	16.211	4.019	0.127 7E-7	15.549	3.959
1/80	0.452 4E-9	15.977	3.998	0.821 4E-9	15.292	3.935
1/160	0.283 1E-10	13.633	3.769	0.537 1E-10	10.375	3.375
1/320	0.207 7E-11	*	*	0.517 7E-11	*	*

参考文献:

[1] 金承日. 解抛物线型方程的高精度显示差分格式[J]. 计算数学,1991,13(1):38-44.

[2] 马明书. 一维抛物线方程的一个新的高精度显示差分格式[J]. 数值计算与计算机应用,2001,22(2):156-160.

[3] 刘利斌,刘焕文. 一维抛物线方程的样条子域精细积分(SSPI)隐格式[J]. 数值计算与计算机应用,2008,29(2):146-152.

[4] 杨国锋,李波,田巧娴,等. 求解一维抛物型方程的一种高精度半显示差分方法[J]. 西北师范大学学报:自然科学版,2008,44(3):8-11.

[5] 詹涌强. 解抛物型方程的一族六点隐式差分格式[J]. 安徽大学学报:自然科学版,2012,36(4):26-29.

[6] Caglar H, Ozer M, Caglar N. The numerical solution of the one-dimensional heat equation by using third degree B-spline functions[J]. Chaos Solitons Fractals, 2008, 38(4): 1197-1201.

[7] Sun Zhizhong. Compact difference schemes for heat equation with Neumann boundary conditions[J]. Numerical Methods Partial Differential Equation, 2009, 25(6): 1320-1341.

[8] Chawla M, Zanaidi M A, Evans D J. Generalized trapezoidal formulas for parabolic equations[J]. Intentional Journal of Computer Mathematics, 1999, 70(3): 429-443.

[9] Sallam S, Karaballi A A. A quartic C^3-spline collocation method for solving second-order initial value problems[J]. Journal of Computational and Applied Mathematics, 1996, 75(2): 295-304.

[10] Liu Huanwen, Liu Libin. An unconditionally stable spline difference scheme of $O(k^2 + h^4)$ for solving the second-order 1D linear hyperbolic equation[J]. Mathematical and Computer Modelling, 2009, 49(9/10): 1985-1993.

[11] Liu Libin, Liu Huanwen. A new fourth-order difference scheme for solving an N-carrier system with Neumann boundary conditions[J]. International Journal of Computer Mathematics, 2011, 88(6): 3553-3564.

(下转第76页)

Journal of Computational Physics, 2002, 178(2): 464 - 497.

[7] Santos C A, Spim J A. A Garcia mathematical modeling and optimization strategies (genetic algorithm and knowledge base) applied to the continuous casting of steel [J]. Engineering Applications of Artificial Intelligence, 2003, 16(5/6): 511 - 527.

[8] 纪振平,马交成,谢植,等. 基于混沌蚁群算法的连铸二冷参数多准则优化 [J]. 东北大学学报:自然科学版,2008,29(6):782 - 785.

[9] 张建立,周晓敏,刘天玉,等. 基于改进粒子群算法的板坯二冷制度优化 [J]. 铸造技术,2007,28(2):248 - 252.

[10] 王杰,毕浩洋. 一种基于粒子群优化的极限学习机 [J]. 郑州大学学报:理学版,2013,45(1):100 - 104.

Multi-objective Model Saving Water and Improving Efficiency in Continuous Casting Secondary Cooling

WANG Hai-ke[1], LI Ji-yun[2], PEI Hong-xing[3]

(1. *Department of Edition, Zhengzhou University, Zhengzhou* 450001, *China*;

2. *Department of Information Engineering, Henan Polytechnic, Zhengzhou* 450046, *China*;

3. *School of Physics and Engineering, Zhengzhou University, Zhengzhou* 450001, *China*)

Abstract: In the new ant colony optimal (ACO) algorithm, node-optimization was used to reduce the time of node-selection and processing costs. The improved ACO algorithm could be enhanced greatly comparing with the traditional ACO. Considering about saving water model and casting speed optimal model, the new multi-objective model consistsed of metallurgical criteria and equipment constraints. The improved ACO algorithm was used to optimize the continuous casting slabs secondary cooling. The method could improve the production efficiency and save secondary cooling water as well as ensure the continuous casting slabs quality.

Key words: continuous casting secondary cooling; water saving model; casting speed model; ant colony optimal algorithm

(上接第40页)

Spline Difference Method for Solving Parabolic Equations with Neumann Boundary Conditions

LIU Rui[1,2], GAO Rui-min[3]

(1. *Department of Science Teaching, Zhengzhou Kindergarten Teacher' College, Zhengzhou* 450000, *China*;

2. *College of Education Administration, Beijing Normal University, Beijing* 100875, *China*;

3. *Department of Basic Course, Henan University of Animal Husbandry and Economy, Zhengzhou* 450044, *China*)

Abstract: Based on the quartic spline function and generalized trapezoidal formulas, a family of implicit difference schemes, including parameter $\theta, \theta \in [0,1]$, for solving parabolic equation with Neumann boundary conditions were constructed. The accuracy of these schemes was second-order in time direction and fourth-order in space direction. If $\theta = 1/3$, the accuracy of this scheme in time direction was improved to third-order. At last, the numerical results showed that our methods were very efficient.

Key words: parabolic equation; quartic spline function; difference scheme; Neumann boundary condition